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Abstract— **In this paper, we study the global stabilization, by means of smooth state feedback, of partially linear composite systems. We show how to compute the stabilizing feedback thanks to a weak Lyapunov function for a nonlinear subsystem instead of a stricte one.**

Keywords— **Nonlinear systems, feedback, global stabilization, Lyapunov function.**

I. Introduction

Many recent papers (see [1], [2], [6] and references therein) addressed the problem of The global stabilization, by means of state feedback, of nonlinear control systems of the form:

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = Ay + Bu \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^p, u \in \mathbb{R}^k, A \in \mathcal{M}_{p,p}(\mathbb{R})$, $B \in \mathcal{M}_{p,k}(\mathbb{R})$ and f is a smooth vector field such that:

(h1) The pair (A, B) is stabilizble.

(h2) The equilibrium $x = 0$ of $\dot{x} = f(x, 0)$ is globally asymptotically stable (G.A.S).

In [6], the authors assumed that the dependence of $f(x, y)$ on y is of the form:

(h3) $f(x,y) = f(x,0) + G(x,y)\,Cy$, with $C \in \mathcal{M}_{k,p}(\mathbb{R})$ and that both C and B are of full rank.

They proved that there exist a matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and a symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ satisfying the following three conditions:

(H1) $P(A + BK) + (A + BK)^T P = -Q$, with Q symmetric positive (^{T}=transpose),

(H2) $(Q^{1/2}, A + BK)$ detectable,

(H3) $B^T P = C$,

if and only if the linear subsytem

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y} = Cy\,, \qquad \tilde{y} \in \mathbb{R}^k \end{cases} \tag{2}$$

is invertible, weakly minimum phase and with CB symetric positive definite.

Using these conditions, they showed that the system (1) is globally asymptotically stabilizable and they gave the stabilizing feedback:

$$u(x,y) = Ky - \frac{1}{2} G(x,y)^T \nabla V(x)$$

where V is a smooth Lyapunov function satisfying:

$$\langle\, \nabla V, f(x,0)\,\rangle\ < 0\ \forall x \in \mathbb{R}^n\,,\ x \neq 0 \tag{3}$$

Notice that the existence of such a strict Lyapunov function V is assured by the condition **(h1)** and the inverse Lyapunov theorem (see [3], [5]). Unfortuantly, there is no systematic method to compute a strict Lyapunov function for a given G.A.S system and it is often easier to constructe a weak Lyapunov function for which the hypotheses of LaSalle's invariance principle (see [4]) are satisfied. As an example one can consider the following system which evolves in $\mathbb{R}^2$ (Liénard's equation):

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -g(x_1) - h(x_1)x_2 \end{cases} \tag{4}$$

where it is assumed that for all $x \neq 0$:

$$xg(x) > 0\,, \quad h(x) > 0$$

and

$$G(x) = \int_0^x g(s)\,ds\ \to \infty \ \text{ as } \ |x| \to \infty$$

For this system, it seems difficult to construct a strict Lyapunov function. However LaSalle's theorem can be applied in an obvious way by taking:

$$V(x_1, x_2) = \frac{1}{2}x_2^2 + G(x_1)$$

In this paper we show that to compute a stabilizing feedback for the system (1) , we do not need to have a strict Lyapunov function for:

$$\dot{x} = f(x,0) \tag{5}$$

We also state that the stabilization procedure is still valid when, in the decomposition **(h3)** of f, the matrix C is of rank $m < k$, provided that CB is of full rank.

II. Notations and definitions

Before stating the main theorem let us introduce the following notations and definitions.

Definition 1: A C^1 scalar function $V : \mathbb{R}^n \to \mathbb{R}$ is a weak Lyapunov function for the system on $\mathbb{R}^n$:

$$\dot{x} = X(x) \tag{6}$$

if V is positive definite proper and satisfies:

$$X.V(x) \leq 0,\ \forall x \in \mathbb{R}^n$$

where $X.V$ is the Lie-derivative of V along the trajectories of the vector field X ($X.V(x) = \langle\, \nabla V(x), X(x)\,\rangle$ where $\langle .,. \rangle$ is the inner product in $\mathbb{R}^n$).

By a *proper function* we mean a function $V : \mathbb{R}^n \to \mathbb{R}$ such that $\{x \in \mathbb{R}^n | V(x) \leq \xi\}$ is compact for each $\xi > 0$. Notice that if the vector field X satisfies the definition 1 then all the trajectories of the system (6) are bounded because of V is proper and its derivative is non positive. For such a vector field, $X_t(.)$ will denote the flow of X defined on $\mathbb{R}^n$. A subset $E \in \mathbb{R}^n$ is said to be X-invariant if for any $x \in E$ on has $X_t(x) \in E$, $\forall t \geq 0$.

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Definition 2: We shall say that the system (6) is of LaSalle-type (L-T) if there exist a weak Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ for (6) such that the largest X-invariant set contained in $E = \{x \in \mathbb{R}^n |\, X.V(x) = 0\}$ is reduced to the origin of $\mathbb{R}^n$.

Remark 1: The system (6) is of (L-T) if and only if it is globally asymptotically stable about the origin of $\mathbb{R}^n$ (see [4]).

Remark 2: It is often easier to find a function V satisfying the definition 2 than one satisfying (3). This is typically the case for mechanical systems for example.

III. Stabilization by LaSalle's invariance principle

Theorem 1: Assume that the pair (A,B) is stabilizable and **(H1)**, **(H2)** and **(H3)** hold. Then if the system (5) is of L-T, so is the closed-loop system (1) with the (stabilizing) feedback:

$$u(x, y) = Ky - G(x, y)^T \nabla V(x) \tag{7}$$

where V is a weak Lyapunov function for (5) as in the definition 2.

Proof: First of all, if the linear subsystem (2) satisfies **(H1)**, **(H2)** and **(H3)**, then it is invertible, weakly minimum phase and with CB symetric positive definite, and it is possible to choose the matrix $K \in \mathcal{M}_{k,p}(\mathbb{R})$ and the symmetric positive definite matrix $P \in \mathcal{M}_{p,p}(\mathbb{R})$ such that:

$$y^T Q\, y = 0 \;\Rightarrow\; C\, y = 0 \tag{8}$$

Indeed, as done in [6], one can assume, without loss of generality, that (2) is in the special ccordinate basis (see [7]):

$$\begin{cases} \dot{y}_{01} = A_{01}y_{01} + A_{11}y_1 \\ \dot{y}_{02} = A_{02}y_{02} + A_{12}y_1 \\ \dot{y}_1 = D_{01}y_{01} + D_{02}y_{02} + D_1y_1 + CBu \\ \tilde{y} = y_1 \end{cases}$$

with A_{01} Hurwitz, $A_{02} + A_{02}^T = 0$, and take:

$$K = (K_{01}, K_{02}, K_1)$$

with:

$$\begin{aligned} K_{01} &= -(CB)^{-1}D_{01} + A_{11}^T P_{01} \\ K_{02} &= (CB)^{-1}D_{02} + A_{12}^T \\ K_1 &= (CB)^{-1}D_1 + \frac{1}{2}\, I \end{aligned}$$

and:

$$P = \begin{pmatrix} P_{01} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & (CB)^{-1} \end{pmatrix}$$

with P_{01} symmetric positive definite such that:

$$P_{01}A_{01} + A_{01}^T P_{01} = -I$$

This particular choice of K and P leads to:

$$Q = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

Hence, $y^T Q\, y = \|y_{01}\|^2 + \|y_1\|^2$ and so:

$$y^T Q\, y = 0 \Rightarrow Cy = y_1 = 0$$

Assume now that the system (5) is of L-T, and set $X(x) = f(x, 0)$, $x \in \mathbb{R}^n$. Let V be a weak Lyapunov function for (5) as in the definition 2 and denote by Ω the largest invariant set by X contained in the locus $E = \{x \in \mathbb{R}^n |\, X.V(x) = 0\}$. By hypotheses $\Omega = \{0\}$. From **(H1)** – **(H3)**, for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, set:

$$Z(x, y) = \begin{pmatrix} f(x, y) \\ h(x, y) \end{pmatrix}$$

where:

$$\begin{aligned} h(x, y) &= Ay + Bu(x, y \\ &= (A + BK)y - BG(x, y)^T \nabla V(x) \end{aligned}$$

and define (see [6]):

$$W(x, y) = V(x) + \frac{1}{2} y^T P\, y$$

W is of class C^1, definite positive and proper, and its derivative along the trajectories of the vector field Z is given by:

$$\begin{aligned} \dot{W}(x, y) &= Z.W(x, y) \\ &= \langle\, Z(x, y)\ ,\ \nabla W(x, y)\, \rangle \\ &= X.V(x) + \langle\, \nabla V(x)\ ,\ G(x, y)Cy\, \rangle \\ &\quad + \frac{1}{2} y^T Q\, y + \langle\, y\ ,\ -PBG(x, y)^T \nabla V(x)\, \rangle \\ &= X.V(x) + \frac{1}{2} y^T Q\, y \\ &\quad + \langle\, y\ ,\ C^T G(x, y)^T \nabla V(x) \\ &\quad - PBG(x, y)^T \nabla V(x)\, \rangle \end{aligned}$$

So, by use of **(H3)** one has:

$$\dot{W}(x, y) = X.V(x) + \frac{1}{2} y^T Q\, y \le 0$$

Notice that all the trajectories of the closed-loop system are bounded because of W is proper and its derivative is non positive. Set:

$$\begin{aligned} \tilde{E} &= \{(x, y) \in \mathbb{R}^{n+p} |\, Z.W(x, y) = 0\} \\ &= \{(x, y) \in \mathbb{R}^{n+p} |\, X.V(x) = 0,\ \text{and } y^T Q\, y = 0\} \end{aligned}$$

According to LaSalle's theorem (see [4] pp 66-67) all the solutions of the closed-loop system tend to $\tilde{\Omega}$ the largest invariant set by Z contained in $\tilde{E}$. in order to prove the theorem 1 let us show that $\tilde{\Omega}$ is the origin of $\mathbb{R}^{n+p}$. By (8), on $\tilde{E}$ the vector field Z is given by:

$$Z(x, y) = \begin{pmatrix} X(x) \\ Y(x, y) \end{pmatrix}$$

where:

$$\begin{aligned} X(x) &= f(x,0) \\ Y(x,y) &= (A+BK)y - BG(x,y)^T\,\nabla V(x) \end{aligned}$$

so that on $\tilde{E}$ the closed-loop system becomes:

$$\begin{cases} \dot{x} = f(x,0) = X(x) \\ \dot{y} = (A+BK)y - BG(x,y)^T\,\nabla V(x) \end{cases}$$

Let $\big(x(t),y(t)\big)$ be a solution of the above system with $\big(x(0),y(0)\big) = (x,y) \in \tilde{\Omega}$. Since $\tilde{\Omega}$ is Z-invariant we have $\big(x(t),y(t)\big) \in \tilde{\Omega}$ for all $t \geq 0$. But one has:

$$\frac{d}{dt}\big(x(t)\big) = X\big(x(t)\big)$$

so that $x(t) = X_t(x)$. Consider now the following set:

$$M = \{x \in \mathbb{R}^n \,|\, \exists\, y \in \mathbb{R}^p, \text{ such that } (x,y) \in \tilde{\Omega}\}$$

If $x \in M$ then $(x,y) \in \tilde{\Omega}$ for some $y \in \mathbb{R}^p$, and for all $t \geq 0$, $(x(t),y(t)) = (X_t(x),y(t)) \in \tilde{\Omega}$ since $\tilde{\Omega}$ is Z-invariant, that implies $X_t(x) \in M$. Then M is X-invariant wich implies that $M \subset \Omega = \{0\}$. So we have shown that:

$$(x,y) \in \tilde{\Omega} \Rightarrow x = 0$$

Since $x(t) = 0$ for all $t \geq 0$, $y(t)$ becomes a solution of $\dot{y} = (A+BK)y$ and $y(t)^T\,Q\,y(t) = 0$ for all $t \geq 0$. Hence, from **(H2)** one deduce that $y(t) = 0$ for all $t \geq 0$ and so $\tilde{\Omega} = \{(0,0)\}$ which completes the proof of theorem 1. ■

Example: Consider the following system evolving in $\mathbb{R}^4$:

$$\begin{cases} \dot{x}_1 = x_2 + (x_1 y_1)^{4/3} y_2 \\ \dot{x}_2 = -x_1^{5/3} - x_1^{4/3} x_2 + (x_1 y_2)^{4/3} y_2 \\ \dot{y}_1 = y_2 \\ \dot{y}_2 = u \end{cases} \tag{9}$$

The subsystem:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^{5/3} - x_1^{4/3} x_2 \end{cases}$$

is of the form (4) and so it is of L-T thanks to the weak Lyapunov function:

$$V(x_1,x_2) = \frac{1}{2}\,x_2^2 + \frac{3}{8}\,x_1^{8/3}$$

Besides, the assumptions **H1** – **H3** hold for the linear subsystem:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = u \\ \tilde{y} = y_2 \end{cases}$$

with $K = (\ -1\ ,\ -\frac{1}{2}\ )$ and $P = I$. Hence the system (9) satisfies the conditions of the theorem 1 and so it is stabilizable thanks to the feedback:

$$u(x_1,x_2,y_1,y_2) = -y_1 - \frac{1}{2}\,y_2 - x_1^3 y_1^{4/3} - x_2(x_1 y_2)^{4/3}$$

Remark 3: Throughout all this work it is supposed that in **(h3)** one has $C \in \mathcal{M}_{k,p}(\mathbb{R})$, so that the linear subsytem (2) has the same number of inputs and outputs. This restriction can be relaxed by assuming that $C \in \mathcal{M}_{m,p}(\mathbb{R})$, $m \leq k$, and (2) is right invertible, weakly minimum phase and with CB of full rank. To make this, notice that, as mentionned in [6], if $m = k$ the assumption CB symetric positive definite can be replaced by CB nonsingular thanks to the use of a static precompensator $u = (CB)^{-1}\tilde{u}$. Then the remark is deduced from the following proposition.

Proposition 1: Assume that $m < k \leq p$ and that both B and CB are of full rank. Then there exists a matrix function $G'(x,y) \in \mathcal{M}_{n,k}(\mathbb{R})$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, and a constant matrix $C' \in \mathcal{M}_{k,p}(\mathbb{R})$ such that $C'B$ is nonsingular and:

$$G(x,y)C = G'(x,y)C'\ , \quad \forall (x,y) \in \mathbb{R}^{n+p}$$

Furthermore, if (2) is weakly minimum phase, so is the linear system:

$$\begin{cases} \dot{y} = Ay + Bu \\ \tilde{y}' = C'y\,, \qquad \tilde{y}' \in \mathbb{R}^k \end{cases} \tag{10}$$

Proof: From the full rank property of B and CB, it is always possible to choose $\tilde{C} \in \mathcal{M}_{k-m,p}(\mathbb{R})$ in such a way that the block-matrix:

$$C' = \begin{pmatrix} C \\ \tilde{C} \end{pmatrix} \tag{11}$$

satisfies $C'B$ nonsingular. For such a choice, and taking $G'(x,y) = \big(\,G(x,y)\ ,\ 0\,\big)$, one has:

$$G(x,y)C = G'(x,y)C'$$

Furthermore, from (11) one can deduce that the zero dynamics of (10) are included in those of (2) which completes the proof of the proposition. ■

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